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QUALITATIVE DISTINCTIONS IN THE SOLUTIONS BASED ON THE PLASTICITY THEORIES WITH THE MOHR–COULOMB YIELD CRITERION

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The following two models of the plasticity theory are considered: the model with the Mohr–Coulomb yield criterion and the classical model of the plasticity theory with a yield criterion independent of the mean stress. The deformation problem of a plastic layer enclosed between two rotating plates is studied.

Key words: *semi-analytical solution, maximum friction, singularity, Mohr–Coulomb criterion, plasticity.*

The plasticity theories based on the Mohr–Coulomb yield criterion are used to describe the motion of granular and loose materials and also deformation of some metallic alloys [1–4]. These models were reviewed in [4]. In the case where the angle of internal friction vanishes, most of these models reduce to the classical theory of plasticity with a yield criterion independent of the mean stress. The solutions of specific problems can, however, diverge from the corresponding solutions based on the classical theory of plasticity, and the solutions obtained by different models may differ qualitatively. In particular, this situation occurs if the law of the maximum friction is used. Alexandrov [5] compared the solutions for two processes (flow of a material through an infinite plane convergent channel and compression of a layer by two parallel rough plates) obtained by two theories of plasticity based on the Mohr–Coulomb yield criterion: Spencer's theory [2] and Hill's theory. The equations of Hill's model, which generalizes to some extent the model of [1], were given in [4]. It was shown in [5] that the solutions corresponding to these models differ qualitatively. However, the drawback of that paper was that the solutions constructed failed to exactly satisfy all boundary conditions. The authors [6] considered the problem that involved the law of the maximum friction and admitted a semi-analytical solution for Spencer's model, which satisfies all boundary conditions exactly. In that paper, the plane strain of a material compressed between rotating rough plates whose surfaces obeyed the law of the maximum friction was studied. The solution of this problem was obtained in [7] using Hill's model, and it was shown that the solution constructed differs qualitatively from the solution given in [6]. In contrast to [6] and [7], it is assumed in the present paper that the plates rotate in such a manner that their opening angle increases. This modification in the formulation of the problem has a substantial effect on the qualitative behavior of the solution for Hill's model.

Figure 1 shows the geometry of the process. It is assumed that there is no outflow at point 0. We introduce polar coordinates r and θ. Owing to symmetry, it suffices to construct the solution in the region $\theta \geqslant 0$. The boundary conditions at the axis of symmetry where $\theta = 0$ are

$$\sigma_{r\theta} = 0; \tag{1}$$

$$v = 0. \tag{2}$$

Here $\sigma_{r\theta}$ is the shear stress in the polar coordinate system. The plates rotate with an angular velocity $\omega > 0$, as is shown in Fig. 1. Therefore, the impermeability condition at the surfaces of the plates becomes

$$v = \omega r \qquad \text{for} \quad \theta = \theta_0. \tag{3}$$

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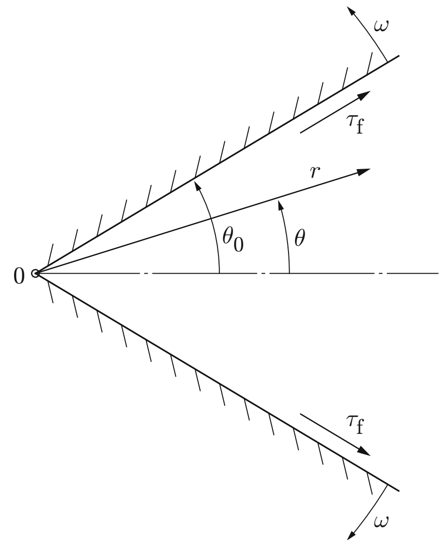

Fig. 1. Geometry of the process.

The opening angle of the plates θ_0 depends on time. Since all the models considered are independent of the strain rate, this angle can be used as an independent variable instead of physical time. At the surfaces of the plates, the law of the maximum friction is specified, which is formulated below for each model of the material. In any case, however, the mode of friction — adhesion or slipping — is unknown in advance. For adhesion, the boundary condition (2) for $\theta = \theta_0$ becomes

$$u = 0. \tag{4}$$

The direction of the specific friction forces (see Fig. 1) is determined from the general considerations. Hence, we have

$$\sigma_{r\theta} > 0. \tag{5}$$

The analytical form of the law of the maximum friction is specific for each model of the material. In the classical theory of plasticity, where the yield criterion is independent of the mean stress, this law takes the form

$$\tau_{\rm f} = \tau_{\rm sh}. \tag{6}$$

Here $\tau_{\rm f}$ are the specific friction forces and $\tau_{\rm sh}$ is the shear yield point. In the case of an ideal plastic material with $\tau_{\rm sh} = \text{const}$, the use of the law of the maximum friction can lead to singular velocity and stress fields [8, 9]. Confining ourselves to plane strain, we reformulate the friction law (6) as follows: the tangent to the friction surface (in fact, to a line for a plane flow) coincides with the characteristic direction of the system of static and kinematic equations of the plasticity theory. In the case of classical plasticity, the characteristics are inclined at angles $\pm\pi/4$ to the first (maximum) principal stress; hence, the friction law (6) for the problem considered is written as

$$\psi = \pi/4 \qquad \text{for} \quad \theta = \theta_0. \tag{7}$$

Here ψ is the angle between the first principal stress and the r axis. According to law (7), the friction surface coincides with a characteristic or an envelope of a family of characteristics. The singular solutions mentioned arise if the friction surface is an envelope of a family of characteristics. It is worth noting that, in the theory of an ideal plastic body, the characteristics of static and kinematic equations coincide.

The friction law in the form of Eq. (7) can be generalized to more complex material models. In particular, for Spencer's model [2], this generalization was proposed in [10, 11]. In this model, as in the theory of an ideal plastic body, the characteristics of static and kinematic equations coincide. It is known [2] that these characteristics are inclined at angles of $\pm(\pi/4 + \varphi/2)$ to the first principal stress (φ is the angle of internal friction). Thus, for Spencer's model, condition (7) should be replaced by

$$\psi = \pi/4 + \varphi/2 \qquad \text{for} \quad \theta = \theta_0. \tag{8}$$

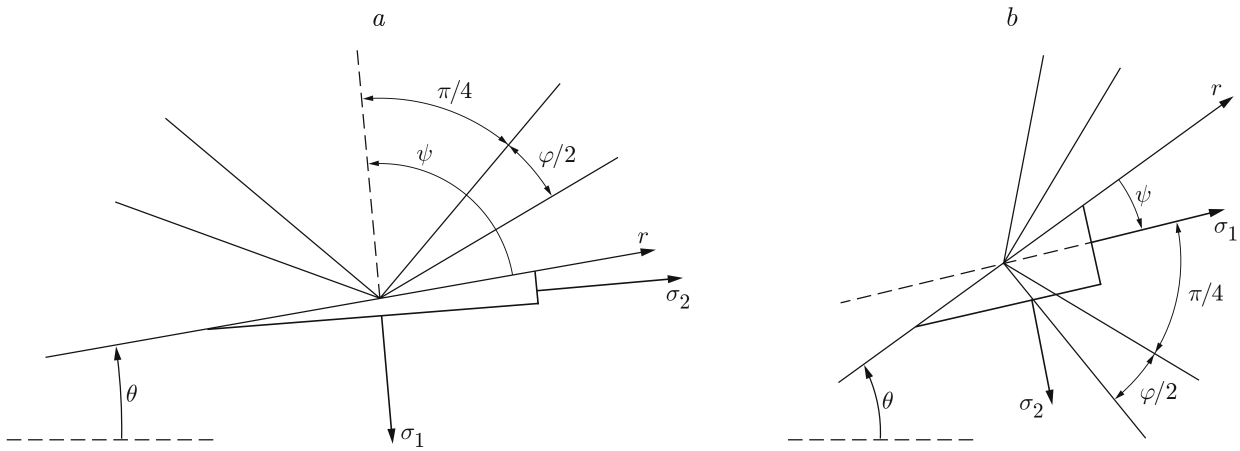

Fig. 2. Directions of the principal stresses and characteristic directions at an arbitrary point for $\omega > 0$ (a) and $\omega < 0$ (b).

It is known [12] that, if the friction surface coincides with an envelope of a family of characteristics, singular velocity fields occur. The asymptotic behavior of the velocity field is the same as in the case of classical plasticity.

For Hill's model, the characteristic directions of static and kinematic equations do not coincide. This property is responsible for the fact that the solutions of the problem considered for $\omega > 0$ and $\omega < 0$ (see Fig. 1) differ qualitatively from each other. The slopes of characteristics of static and kinematic equations with respect to the direction of the first principal stress are determined by angles $\pm(\pi/4 + \varphi/2)$ and $\pm\pi/4$, respectively. Figure 2 shows the directions of the principal stresses σ_1 and σ_2, which acquire this form owing to condition (5), and the characteristic directions at an arbitrary point of the deformed region for $\omega > 0$. As the axis of symmetry where $\theta = 0$ is approached, the direction of the stress σ_1 becomes orthogonal to the r axis. Hence, we have

$$\psi = \pi/2 \qquad \text{for} \quad \theta = 0. \tag{9}$$

This condition is equivalent to condition (1). As the friction surface where $\theta = \theta_0$ is approached, the angle ψ decreases and, as can be seen from Fig. 2, only the characteristic direction of the system of static equations can coincide with the friction surface. Thus, the law of the maximum friction has the form (8).

Figure 2b shows the directions of the principal stresses, which acquire this form owing to the condition $\sigma_{r\theta} < 0$, and the characteristic directions for a material compressed between two plates ($\omega < 0$) with the satisfied equations of Hill's model considered in [7]. In contrast to Eq. (9), the angle ψ tends to zero as the axis of symmetry is approached. Moreover, the angle ψ decreases as the friction surface is approached, so that $\psi \leqslant 0$ in the deformed region. Then, as one can see from Fig. 2b, only the characteristic direction of the system of kinematic equations can coincide with the friction surface. Owing to this fact, the law of the maximum friction was written in [7] as $\psi = -\pi/4$ for $\theta = \theta_0$. For $\omega > 0$, this law would have taken the form (7). As it follows from the solution obtained below, this necessary difference in the formulation of the law of the maximum friction leads to qualitatively different solutions. It is worth noting that this discrepancy does not occur in the classical theory of plasticity and Spencer's model since the characteristic directions for static and kinematic equations coincide. In these cases, a change of sign of ω leads only to a change of sign of the limiting value of ψ on the friction surface.

For a plane flow, the system of equations of Hill's model is written in polar coordinates as

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\,\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0, \qquad \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\,\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{2}{r}\,\sigma_{r\theta} = 0; \tag{10}$$

$$(\sigma_{rr} + \sigma_{\theta\theta})\sin\varphi + [(\sigma_{rr} - \sigma_{\theta\theta})^2 + 4\sigma_{r\theta}^2]^{1/2} = 2k\cos\varphi; \tag{11}$$

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\,\frac{\partial v}{\partial \theta} = 0; \tag{12}$$

$$\sin 2\psi\Big(\frac{\partial u}{\partial r} - \frac{u}{r} - \frac{1}{r}\,\frac{\partial v}{\partial \theta}\Big) - \cos 2\psi\Big(\frac{1}{r}\,\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}\Big) = 0, \tag{13}$$

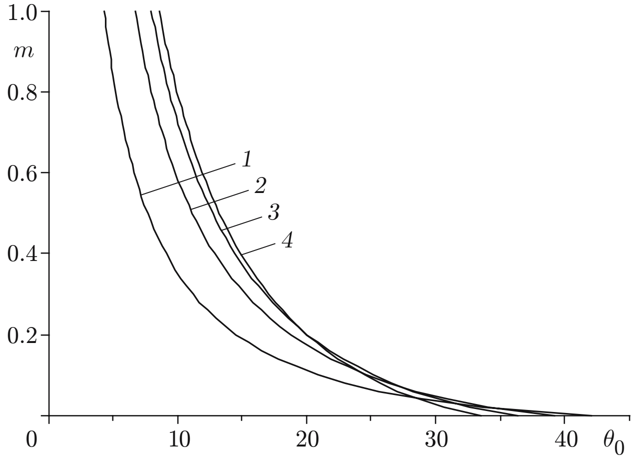

Fig. 3. The value of m versus the opening angle of the plates θ_0 for $\varphi = 0.1$ (1), 0.2 (2), 0.3 (3), 0.4 (4).

where σ_{rr} and $\sigma_{\theta\theta}$ are the normal stresses, v and u are the circumferential and radial velocities in the polar coordinate system, respectively, and $k = \text{const}$ is the adhesion coefficient. In the system given above, Eqs. (10) are the equations of equilibrium, Eq. (11) is the Mohr–Coulomb yield criterion, Eq. (12) is the incompressibility equation, and Eq. (13) is the condition of coaxial stress and strain-rate tensors. For $\varphi = 0$, system (10)–(13) reduces to equations of the plane problem of the theory of ideal plasticity with $k \equiv \tau_{\rm sh}$. For $k = 0$, system (10)–(13) reduces to equations of the model considered in [1]. The general solution of the static equations (10) and (11), in which ψ is assumed to be independent of r, is [7]

$$\frac{\sigma_{rr}}{k} = \cot\,\varphi - \frac{r^m\Phi(\psi,\theta_0)}{\sin\varphi}\,(1-\cos 2\psi\sin\varphi),$$

$$\frac{\sigma_{\theta\theta}}{k} = \cot\,\varphi - \frac{r^m\Phi(\psi,\theta_0)}{\sin\varphi}\,(1+\cos 2\psi\sin\varphi), \tag{14}$$

$$\frac{\sigma_{r\theta}}{k} = r^m\Phi(\psi,\theta_0)\sin 2\psi, \qquad \Phi(\psi,\theta_0)\geqslant 0;$$

$$\Phi = \Phi_0(m\cos^2\varphi - 2\sin^2\varphi - 2\sin\varphi\cos 2\psi)^{m/2}; \tag{15}$$

$$\frac{\partial\psi}{\partial\theta} = \frac{m\cos^2\varphi - 2\sin^2\varphi - 2\cos 2\psi\sin\varphi}{2\sin\varphi\,(\sin\varphi+\cos 2\psi)}. \tag{16}$$

However, the particular solution for ψ, which serves to determine the constant m, differs from that obtained in [7] because of the boundary conditions (8) and (9). Integrating Eq. (16) with allowance for Eq. (9), we obtain

$$\theta = 2\sin\varphi\int\limits_{\pi/2}^{\psi}\frac{\sin\varphi+\cos 2z}{m\cos^2\varphi - 2\sin^2\varphi - 2\sin\varphi\cos 2z}\,dz. \tag{17}$$

Substituting Eq. (8) into Eq. (17), we obtain the equation for m:

$$\theta_0 = 2\sin\varphi\int\limits_{\pi/2}^{\pi/4+\varphi/2}\frac{\sin\varphi+\cos 2z}{m\cos^2\varphi - 2\sin^2\varphi - 2\sin\varphi\cos 2z}\,dz. \tag{18}$$

Figure 3 shows the dependence of m on θ_0 obtained from the solution of Eq. (18) for different values of the angle of internal friction. It should be noted that, if $m < 0$ and the absolute value of this quantity is reasonably small, a nonintegrable singularity appears in Eq. (18). Thus, the angle θ_0 for $m = 0$ (Fig. 3) reaches the limiting values for which the solution obtained occurs. For large values of the angle θ_0, one should introduce a rigid zone adjacent to the friction surface, as it was done in [6]. In the present paper, however, it is assumed that the angle θ_0 is smaller than the limiting value. It follows from Eq. (17) that $d\theta/d\psi = 0$ for $\psi = \pi/4 + \varphi/2$. Evaluating the second derivative $d^2\theta/d\psi^2$, one infers that its sign at this point is opposite to the sign of m. In this case, as it follows from Fig. 3, the condition $d^2\theta/d\psi^2 < 0$ holds for $\psi = \pi/4 + \varphi/2$ and the function $\theta(\psi)$ attains a maximum at this point. Consequently, the continuation of the solution of Eq. (17) into the region $\psi < \pi/4 + \varphi/2$ makes no physical sense, and Eq. (8) is the only possible law of the maximum friction. Determining m from the solution of Eq. (18) and the distribution $\psi(\theta)$ from Eq. (17), one obtains the stresses in the deformed region from Eqs. (14) and (15) up to a constant Φ_0, which remains indeterminate as in other problems of this class.

We write the velocity field as

$$u = \frac{\omega r}{2}\frac{dg}{d\theta}, \qquad v = -\omega r g. \tag{19}$$

Here g is an arbitrary function independent of r. The velocity field (19) satisfies the incompressibility equation (12) for any function g. With allowance for Eqs. (8) and (9), the boundary conditions (2) and (3) become

$$g = 0 \qquad \text{for} \quad \psi = \pi/2; \tag{20}$$

$$g = -1 \qquad \text{for} \quad \psi = \pi/4 + \varphi/2. \tag{21}$$

Substitution of Eq. (19) into Eq. (13) yields

$$\cos 2\psi \frac{d^2 g}{d\theta^2} - 2\sin 2\psi \frac{dg}{d\theta} = 0. \tag{22}$$

We introduce a new function

$$G = \frac{dg}{d\theta} = \frac{dg}{d\psi}\frac{d\psi}{d\theta}. \tag{23}$$

Inserting Eq. (23) into Eq. (22), using Eq. (16), and integrating the resultant equation, we obtain

$$\ln\frac{G}{G_0} = 4\sin\varphi \int\limits_{\pi/4+\varphi/2}^{\psi} \frac{\tan\, 2z\,(\sin\varphi + \cos 2z)}{m\cos^2\varphi - 2\sin^2\varphi - 2\sin\varphi\cos 2z}\, dz, \tag{24}$$

where G_0 is an arbitrary function of θ_0. It follows from Eqs. (19), (23), and (24) that the quantity G_0 is proportional to the slip velocity of the material at the surface of the plates. Substituting Eq. (24) into Eq. (23), using Eq. (16), and integrating, we obtain

$$g = 2G_0\sin\varphi \int\limits_{\pi/2}^{\psi} \Big[\frac{\sin\varphi + \cos 2x}{m\cos^2\varphi - 2\sin^2\varphi - 2\sin\varphi\cos 2x}$$

$$\times \exp\Big(4\sin\varphi \int\limits_{\pi/4+\varphi/2}^{x} \frac{\tan\, 2z\,(\sin\varphi + \cos 2z)}{m\cos^2\varphi - 2\sin^2\varphi - 2\sin\varphi\cos 2z}\, dz\Big)\Big]\, dx. \tag{25}$$

Solution (25) satisfies the boundary condition (20). The quantity G_0 is determined from Eq. (25) and boundary condition (21) as

$$-1 = 2G_0\sin\varphi \int\limits_{\pi/2}^{\pi/4+\varphi/2} \Big[\frac{\sin\varphi + \cos 2x}{m\cos^2\varphi - 2\sin^2\varphi - 2\sin\varphi\cos 2x}$$

$$\times \exp\Big(4\sin\varphi \int\limits_{\pi/4+\varphi/2}^{x} \frac{\tan\, 2z\,(\sin\varphi + \cos 2z)}{m\cos^2\varphi - 2\sin^2\varphi - 2\sin\varphi\cos 2z}\, dz\Big)\Big]\, dx.$$

Numerical integration of this equation with the use of the solution of Eq. (18) yields the dependence of G_0 on θ_0 plotted in Fig. 4 for different values of the angle φ. The dependences of G and g on θ, which completely characterize

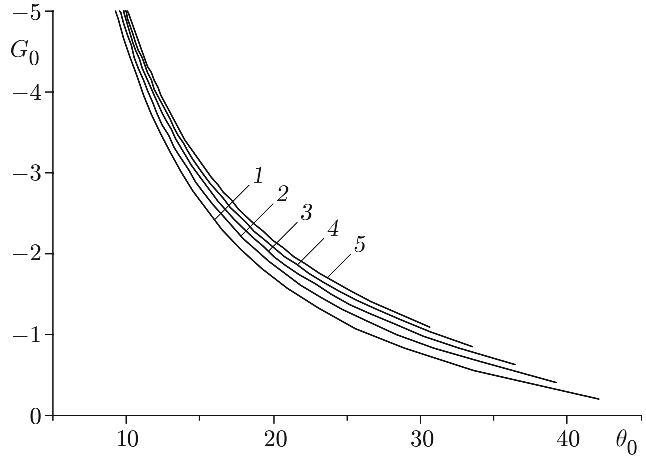

Fig. 4. The quantity G_0 versus the opening angle of the plates θ_0 for $\varphi = 0.1$ (1), 0.2 (2), 0.3 (3), 0.4 (4), and 0.5 (5).

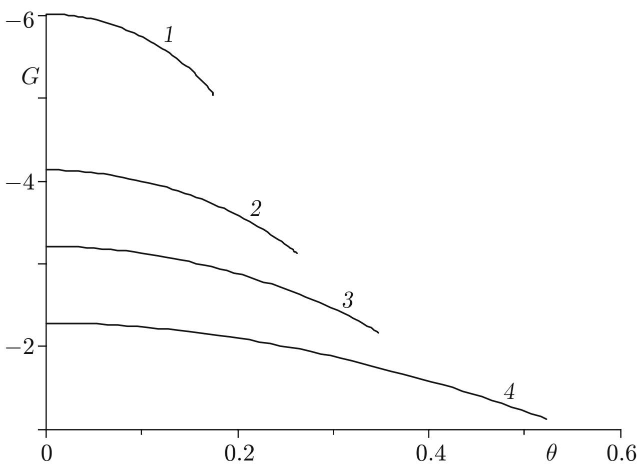

Fig. 5

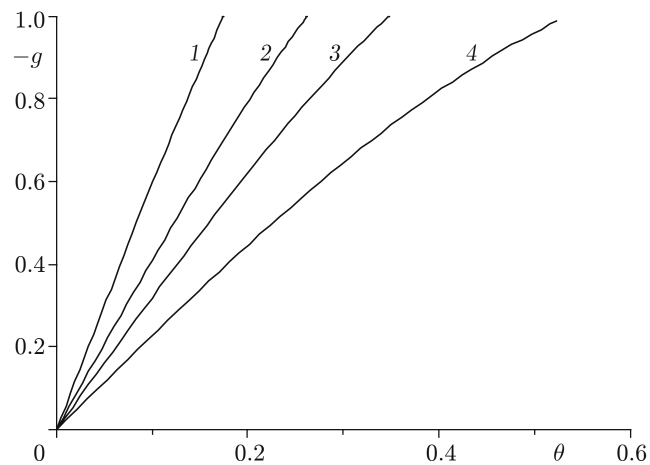

Fig. 6

Fig. 5. Function G characterizing the radial-velocity distribution versus the angle θ for $\varphi = 0.5$ for different stages of the process: $\theta_0 = 10^\circ$ (1), 15° (2), 20° (3), and 30° (4).

Fig. 6. Function g characterizing the circumferential-velocity distribution versus the angle θ for $\varphi = 0.5$ for different stages of the process: $\theta_0 = 10^\circ$ (1), 15° (2), 20° (3), and 30° (4).

the velocity distribution across the layer by virtue of Eq. (19), are shown in Figs. 5 and 6, respectively, for $\varphi = 0.5$ (this value of the internal-friction angle is typical of many granular materials) for several stages of the process (several values of the angle θ_0). Since $G_0 \neq 0$, the adhesion condition (4) is not satisfied. This is one of the qualitative distinctions of the present solution from the solution obtained in [7], in which the law of the maximum friction written in stresses leads to satisfaction of condition (4) in velocities. At the same time, slipping occurs in the solution based on Spencer's model [6] if the law of the maximum friction is used. In this case, the velocity field is singular: in polar coordinates, the shear-strain rate tends to infinity as the friction surface is approached. For some parameters of the process, the velocity field is singular in the solution of [7] as well. We show that the shear-strain rate in the solution obtained is finite as the friction surface is approached. Expanding the integrand in (24) into a series in the neighborhood of the point $\psi = \pi/4 + \varphi/2$ and integrating, we obtain

$$G = G_0 \exp\Big[\frac{4}{m}\Big(\psi - \frac{\pi}{4} - \frac{\varphi}{2}\Big)^2\Big] \tag{26}$$

(the higher-order terms are omitted). The shear-strain rate $\xi_{r\theta}$ may tend to infinity if only $d^2g/d\theta^2 = dG/d\theta \to \infty$

as $\psi \to \pi/4+\varphi/2$. Expanding the right side of Eq. (16) into a series in the neighborhood of the point $\psi = \pi/4+\varphi/2$, from Eqs. (26) and (16), we obtain

$$\frac{dG}{d\theta} = -2G_0 \cot\varphi \exp\Big[\frac{4}{m}\Big(\psi - \frac{\pi}{4} - \frac{\varphi}{2}\Big)^2\Big] \tag{27}$$

as $\psi \to \pi/4 + \varphi/2$. Thus, it follows from Eq. (27) that the shear-strain rate is limited near the friction surface. However, the stress state is singular since the derivatives of some components of the stress tensor with respect to θ tend to infinity as the friction surface is approached. We consider, for example, the component σ_{rr}. Differentiating Eq. (14) with respect to θ and replacing ψ by $\pi/4 + \varphi/2$ everywhere except for the factor $d\psi/d\theta$, we obtain

$$\frac{1}{k}\frac{\partial\sigma_{rr}}{\partial\theta} = -4\Phi_0 r^m m^{m/2} \cos^{m-1}\varphi\,\frac{d\psi}{d\theta}. \tag{28}$$

Expanding the right side of Eq. (16) into a series as $\psi \to \pi/4+\varphi/2$, we write this equation near the friction surface as

$$\frac{\partial\psi}{\partial\theta} = -\frac{m\cos\varphi}{4\sin\varphi\,(\psi - \pi/4 - \varphi/2)}. \tag{29}$$

Integration of this equation with allowance for the boundary condition $\psi = \pi/4 + \varphi/2$ for $\theta = \theta_0$ yields

$$\psi - \frac{\pi}{4} - \frac{\varphi}{2} = \Big(\frac{m}{2\tan\varphi}\Big)^{1/2}(\theta_0 - \theta)^{1/2}. \tag{30}$$

Substituting Eqs. (29) and (30) into Eq. (28), we obtain

$$\frac{1}{k}\,\frac{\partial\sigma_{rr}}{\partial\theta} = \frac{\Phi_0 r^m m^{(m/2+1/2)}(2\tan\varphi)^{1/2}\cos^m\varphi}{\sin\varphi\,(\theta_0-\theta)^{1/2}} + o[(\theta_0-\theta)^{-1/2}]. \tag{31}$$

Using the same reasoning, we find that the derivatives $\partial\sigma_{\theta\theta}/\partial\theta$ and $\partial\sigma_{r\theta}/\partial\theta$ tend to a finite limit on the friction surface. It is worth noting that, for Spencer's model, the behavior of the stress σ_{rr} near the surface of the maximum friction discussed in [6] is the same as that predicted by Eq. (31).

Let us construct a solution using the theory of an ideal plastic body. It is necessary to set $\varphi = 0$ in Eq. (11). The solution for stresses was obtained in [13] and is written in the notation adopted above as

$$\begin{gathered}\sigma_{rr} = k(\sigma + \cos 2\psi), \qquad \sigma_{\theta\theta} = k(\sigma - \cos 2\psi), \qquad \sigma_{r\theta} = k\sin 2\psi,\\ \sigma = -A\ln r - (A/2)\ln((A - 2\cos 2\psi)/B).\end{gathered} \tag{32}$$

Here A and B are arbitrary constants and ψ is a function of the angle θ, which is determined from the equation

$$\frac{\partial\psi}{\partial\theta} = \frac{A - 2\cos 2\psi}{2\cos 2\psi} \tag{33}$$

and has the form

$$\theta = 2\int\limits_{\pi/2}^{\psi} \frac{\cos 2x}{A - \cos 2x}\,dx. \tag{34}$$

The lower limit of integration is chosen in such a manner that condition (9) is satisfied on the axis of symmetry. Condition (7) should be satisfied on the friction surface. It follows from Eq. (34) that

$$\theta_0 = 2\int\limits_{\pi/2}^{\pi/4} \frac{\cos 2x}{A - \cos 2x}\,dx. \tag{35}$$

This equation determines the quantity A for a specified value of θ_0. Equation (35) can be solved by a numerical method, which is not required, however, for the goals of the present paper.

Equations (12) and (13) do not depend on φ. Therefore, the representation of the velocity field (19), boundary conditions (20) and (21), Eq. (22), and substitution (23) are valid. Since the dependence $\psi(\theta)$ is given here by Eq. (34), however, we obtain the following solution instead of solution (24):

$$G/G_0 = (A - 2\cos 2\psi)/A. \tag{36}$$

Here G_0 has the same physical meaning as in solution (24). Substituting Eq. (36) into Eq. (23), taking into account Eq. (33), and integrating with allowance for the boundary conditions (20) and (21), we find that $G_0 = -A$ and

$$g = -\sin 2\psi. \tag{37}$$

Expanding the right side of Eq. (33) into a series in the neighborhood of the point $\psi = \pi/4$ and integrating with allowance for the condition $\psi = \pi/4$ for $\theta = \theta_0$, we obtain

$$\psi - \pi/4 = \sqrt{A/2}\,(\theta_0 - \theta)^{1/2} + o[(\theta_0 - \theta)^{1/2}]. \tag{38}$$

This expression is similar to relation (30). We evaluate the derivative $\partial u/\partial\theta$. Using Eqs. (19), (37), and (38), we write this derivative near the friction surface as

$$\frac{\partial u}{\partial\theta} = \omega r\sqrt{\frac{A}{2}}\,(\theta_0 - \theta)^{-1/2} + o[(\theta_0 - \theta)^{-1/2}]. \tag{39}$$

Since $\partial u/\partial\theta$ enters the expression for the shear-strain rate $\xi_{r\theta}$, it follows from Eq. (39) that $\xi_{r\theta} \to \infty$ as $\theta \to \theta_0$. This is the principal distinction from the solution for $\varphi \neq 0$. In the latter case, relation (27) implies that $\xi_{r\theta}$ tends to a finite limit as $\theta \to \theta_0$. In both cases, however, the stresses behave identically. Calculating the derivative $\partial\sigma_{rr}/\partial\theta$ with the use of (32), (33), and (38), we obtain

$$\frac{1}{k}\,\frac{\partial\sigma_{rr}}{\partial\theta} = \frac{\sqrt{2A}}{(\theta_0 - \theta)^{1/2}} + o[(\theta_0 - \theta)^{-1/2}],$$

which corresponds to Eq. (31). Using the same equations, one can calculate the derivatives $\partial\sigma_{\theta\theta}/\partial\theta$ and $\partial\sigma_{r\theta}/\partial\theta$ and show that they tend to a finite limit as $\theta \to \theta_0$.

In summary, we can note that the solution obtained in [6] based on Spencer's model has the same qualitative specific features as the solution obtained for $\varphi = 0$.

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REFERENCES

1. A. Yu. Ishlinskii, "Plane motion of sand," *Ukr. Mat. Zh.*, **6**, No. 4, 430–441 (1954).
2. A. J. M. Spencer, "A theory of the kinematics of ideal soils under plane strain conditions," *J. Mech. Phys. Solids*, **12**, 337–351 (1964).
3. W. A. Spitzig, R. J. Sober, and O. Richmond, "The effect of hydrostatic pressure on the deformation behavior of maraging and HY-80 steels and its implications for plasticity theory," *Met. Trans.*, **7A**, No. 11, 1703–1710 (1976).
4. J. Ostrowska-Maciejewska and D. Harris, "Three-dimensional constitutive equations for rigid/perfectly plastic granular materials," *Math. Proc. Cambridge Philos. Soc.*, **108**, 153–169 (1990).
5. S. Alexandrov, "Comparison of double-shearing and coaxial models of pressure-dependent plastic flow at frictional boundaries," *Trans. ASME, J. Appl. Mech.*, **70**, No. 2, 212–219 (2003).
6. S. E. Alexandrov and E. A. Lyamina, "Compression of a plastic material sensitive to the mean stress by rotating plates," *Izv. Ross. Akad. Nauk, Mekh. Tverd. Tela*, No. 6, 50–60 (2003).
7. S. E. Alexandrov, "Singular solutions in one model of plasticity with the yield criterion dependent on the mean stress," in: *Problems of Mechanics* (collected scientific papers) [in Russian], Fizmatlit, Moscow (2003), pp. 77–86.
8. S. E. Alexandrov, "Velocity field near the surface of their discontinuity for an arbitrary flow of an ideal rigid-plastic material," *Izv. Ross. Akad. Nauk, Mekh. Tverd. Tela*, No. 5, 116–122 (1995).
9. S. Alexandrov and O. Richmond, "Singular plastic flow fields near surfaces of maximum friction stress," *Int. J. Non-Linear Mech.*, **36**, No. 1, 1–11 (2001).
10. C. S. Pemberton, "Flow of imponderable granular materials in wedge-shaped channels," *J. Mech. Phys. Solids*, **13**, 351–360 (1965).
11. E. A. Marshall, "The compression of a slab of ideal soil between rough plates," *Acta Mech.*, **3**, 82–92 (1967).
12. S. E. Alexandrov and E. A. Lyamina, "Singular solutions for a plane flow of materials sensitive to the mean stress," *Dokl. Ross. Akad. Nauk*, **383**, No. 4, 492–495 (2002).
13. V. V. Sokolovskii, "Plane and axisymmetric equilibrium of a plastic mass between rigid walls," *Prikl. Mat. Mekh.*, **14**, 75–92 (1950).